Solutions to HW Problem Set #3

8.1 (a) In conductor, force per unit volume is $\vec{j}\times\vec{B}_c = \mu_c \vec{j}_c \times \vec{H}_c$

From Maxwell's Eqn, neglecting displacement current $\vec{j}_c = \nabla\times\vec{H}_c$

x, y in plane of surface
z normal to surface

$\therefore$ Force per unit area normal to surface is $\mu_c\int\left((\vec{\nabla}\times\vec{H}_c)\times\vec{H}_c\right)\cdot\vec{n}\, d\xi$

(integrating over depth into conductor ξ). Neglect derivatives $\parallel$ to surface

Then $\left[(\vec{\nabla}\times\vec{H}_c)\times\vec{H}_c\right]\cdot\vec{n} = -\left(\frac{\partial H_x}{\partial z}\right)H_x - \left(\frac{\partial H_y}{\partial z}\right)H_y = -\frac{1}{2}\frac{\partial}{\partial z}\left(H_{\parallel c}^2\right)$

$$\therefore \mu_c\int\left((\vec{\nabla}\times\vec{H}_c)\times\vec{H}_c\right)\cdot\vec{n}\, d\xi = -\frac{\mu_c}{2}\Big[H^2_{\parallel,c\ \text{surface}} - \underbrace{H^2_{\parallel,c\ \text{interior}}}_{0}\Big]$$

$$= -\frac{\mu_c}{2} H^2_{\parallel,c\ \text{surface}}$$

$\therefore H_{\parallel,c} = H_{\parallel\,\text{surface}}\cos(\omega t+\phi)$ $\therefore$ time average of $H^2_{\parallel,c\ \text{surface}} = \frac{1}{2}|H_\parallel|^2$

$\therefore$ Time average of $\vec{f} = -\frac{\mu_c}{4}|H_\parallel|^2\,\vec{n}$

(b) Different magnetic permeability μ outside the surface would affect only normal component of magnetic field which does not enter above expression. No free charges, displacement current negligible so no electric forces

(c) If $H_\parallel = \sum_i H_{i\parallel}\cos(\omega_i t+\phi_i)$ (random phases)

$\langle |H_\parallel|^2\rangle = \frac{1}{2}\sum_i H_{i\parallel}^2$ $\qquad \sum_i H_{i,\parallel}^2 = 2\langle |H_\parallel|^2\rangle$ $\qquad \vec{f} = -\frac{\mu_c}{4}\sum_i (H_i^\parallel)^2\,\vec{n}$

$\therefore$ result follows.

8.2 (a) For TEM mode, $\nabla\times\vec{E}_{TEM} = 0$, $\nabla\cdot\vec{E}_{TEM} = 0$ in space between conductors.

$\therefore \vec{E}_{TEM}$ satisfies electrostatic problem with cylindrical symmetry (apart from $e^{ikz-\omega t}$ factor)

ie $\quad \vec{E}_{TEM} = \frac{C}{r}\hat{r}$ (1) ($\hat{r}$ = Radial direction) $\quad$ (C = Constant)

(Peak value) $\rightarrow$ $\vec{H} = \frac{\sqrt{\mu\epsilon}}{\mu}\,\hat{z}\times\vec{E}_{TEM} = \sqrt{\frac{\epsilon}{\mu}}\,\frac{C}{r}\,\hat{\theta}$ (2) ($\hat{\theta}$ = azimuthal direction)

$\therefore$ C can be obtained from value of H at surface of inner cylinder ($r=a$)

(Peak value!) $H_0 = \sqrt{\frac{\epsilon}{\mu}}\frac{C}{a}$, ie $C = \sqrt{\frac{\mu}{\epsilon}}\,a\,H_0$ (3)

Time-averaged $\vec{S} = \frac{1}{2}(\vec{E}\times\vec{H}^*) = \frac{1}{2}\left(\vec{E}\times\frac{\sqrt{\mu\epsilon}}{\mu}(\hat{z}\times\vec{E}^*)\right) = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E|^2$ along $\hat{z}$

8.2 (a) contd.

$\therefore$ rate of power flow $P = \int S\, dA = \int_a^b S\; 2\pi r\, dr$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} \int_a^b |E|^2\, 2\pi r\, dr = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\, C^2\, 2\pi \int_a^b \frac{1}{r}\, dr \quad (3)$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\left(\frac{\mu}{\epsilon}\right) a^2 |H_0|^2\, 2\pi \ln\left(\frac{b}{a}\right) = \sqrt{\frac{\mu}{\epsilon}}\, \pi a^2\, |H_0^2| \ln\left(\frac{b}{a}\right)$$

(b) $P(z) = P_0 e^{-2\gamma z}$

$$\gamma = -\frac{1}{2P}\frac{dP}{dz} \qquad \text{(see Eqs. (8.56) + (8.57) of Book)}$$

$$= +\frac{1}{2P}\frac{1}{2\sigma\delta}\oint_c (\hat{n}\times\vec{H})^2\, dl \qquad \text{(see Eqn (8.58) of book)}$$

where $\oint_c$ means around circumference of conductors (inner + outer)

Contribution from inner conductor of $\oint_c (\hat{n}\times\vec{H})^2 dl = \left(\frac{\epsilon}{\mu}\right)\frac{C^2}{a^2}\, 2\pi a$ from (2) above

$$= \frac{\epsilon}{\mu}\cdot 2\pi C^2 \cdot \frac{1}{a}$$

Similarly contribution from outer conductor $= \frac{\epsilon}{\mu}\, 2\pi C^2 \left(\frac{1}{b}\right)$

$$\therefore \oint_c (\hat{n}\times\vec{H})^2 dl = \frac{\epsilon}{\mu}\cdot 2\pi C^2 \left(\frac{1}{a}+\frac{1}{b}\right)$$

$$\therefore \gamma = +\frac{1}{2P}\frac{1}{2\sigma\delta}\frac{\epsilon}{\mu}\, 2\pi\, C^2 \left(\frac{1}{a}+\frac{1}{b}\right)$$

But $P = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\, C^2\, 2\pi \ln\left(\frac{b}{a}\right)$ (from (3) above)

$$\therefore \gamma = \frac{1}{2\sigma\delta}\frac{\left(\frac{\epsilon}{\mu}\right) 2\pi C^2 \left(\frac{1}{a}+\frac{1}{b}\right)}{\sqrt{\frac{\epsilon}{\mu}}\, 2\pi C^2 \ln\left(\frac{b}{a}\right)} = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

(since $\vec{E}$ is negligible inside conductors, we neglect contribution to $\vec{S}$ from inside conductors)

(c) Voltage between cylinders is $\Delta V = \int_a^b (\vec{E}_{TEM}\cdot\hat{r})\, dr$

$$= C \int_a^b \frac{1}{r}\, dr = C \ln\left(\frac{b}{a}\right)$$

* Total current flowing in inner cylinder = Surface Current (integrating over skin depth) $\times\, 2\pi a$

$$\therefore I = |\hat{n}\times\vec{H}_\parallel|\, 2\pi a \quad \text{(from Eq. (8.14) of book)}$$

$$= \sqrt{\frac{\epsilon}{\mu}}\frac{C}{a}\cdot 2\pi a = \sqrt{\frac{\epsilon}{\mu}}\, 2\pi C, \quad \text{so } Z_0 = \frac{\Delta V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \ln\left(\frac{b}{a}\right)$$

(from (2))

8.2 (b) contd

$$I = |\vec{n}\times H_{\parallel}|\, 2\pi a \quad \text{(from Eq. 8.14 of book)} \qquad (4)$$

$$= \sqrt{\frac{\epsilon}{\mu}}\,\frac{C}{a}\,2\pi a = \sqrt{\frac{\epsilon}{\mu}}\,2\pi C \qquad (4a)$$

$$\therefore Z_0 = \frac{\Delta V}{I} = \frac{\cancel{C}\ln\left(\frac{b}{a}\right)}{\sqrt{\frac{\epsilon}{\mu}}\,2\pi\,\cancel{C}} = \sqrt{\frac{\mu}{\epsilon}}\,\frac{1}{2\pi}\ln\left(\frac{b}{a}\right)$$

(d) Ohmic Loss per unit length (averaged over time) = W, say

$$W = \frac{1}{2}I^2 R \qquad (I = \text{peak current, given by (4) above})$$

$$\therefore R = \frac{2W}{I^2}$$

Time-averaged Ohmic Loss per unit volume in conductor $= \frac{1}{2\sigma}|\vec{J}|^2$

where $\vec{J} = \frac{1}{\delta}(1-i)(\vec{n}\times\vec{H}_{\parallel})e^{-\xi(1-i)/\delta}$ (Eq. 8.13 of book)

(peak value)

$$|\vec{J}|^2 = \frac{2}{\delta^2}\left|(\vec{n}\times\vec{H}_{\parallel})^2\right| e^{-2\xi/\delta}$$

For inner cylinder, integrating over ξ & circumference (2πa)

Time-averaged Ohmic Loss in ~~cylinder~~ Inner cylinder $= \frac{1}{2\sigma}\,\frac{2}{\delta^2}\,\frac{C^2}{a^2}\,\frac{\delta}{2}(2\pi a)\left(\frac{\epsilon}{\mu}\right)$

(from Eq. (2))

$$= \frac{1}{2\sigma\delta}\left(\frac{\epsilon}{\mu}\right)2\pi C^2\,\frac{1}{a}$$

$\therefore$ summing over both cylinders, $W = \frac{1}{2\sigma\delta}\left(\frac{\epsilon}{\mu}\right)2\pi C^2\left(\frac{1}{a}+\frac{1}{b}\right)$

$$\therefore R = \frac{2W}{I^2} = \frac{\frac{1}{\sigma\delta}\left(\frac{\epsilon}{\mu}\right)2\pi C^2\left(\frac{1}{a}+\frac{1}{b}\right)}{\left(\frac{\epsilon}{\mu}\right)(2\pi)^2C^2}$$

from Eq 4(a) above

$$\therefore R = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$

<u>Inductance</u> per unit length obtained from $L = \frac{1}{I^2}\int\frac{\vec{B}\cdot\vec{B}}{\mu}d^3\vec{x}$ integrated over volume of unit length

$$= \frac{1}{I^2}\int\frac{\vec{B}\cdot\vec{B}}{\mu}dA$$

(a) Consider space between conductors

$$\int\frac{\vec{B}\cdot\vec{B}\,dA}{\mu} = \mu\int_a^b |H|^2\,2\pi r\,dr \qquad H = \sqrt{\frac{\epsilon}{\mu}}\,\frac{C}{r}$$

$$= \mu\left(\frac{\epsilon}{\mu}\right)\cdot C^2\,2\pi\int_a^b\frac{r}{r^2}\,dr = \epsilon C^2\,2\pi\ln(b/a)$$

$I^2 = \left(\frac{\epsilon}{\mu}\right)(2\pi)^2C^2$ (from (4a) above) $\therefore L_1 = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$ from space between conductors

Add contribution from inner conductor $\frac{1}{I^2}\int \frac{\vec{B}\cdot\vec{B}}{\mu_c}\,dA = \frac{\mu_c}{I^2}\int_{\substack{\text{inner}\\ \text{conductor}}} |H_c|^2 dA$

$$= \frac{1}{I^2}\mu_c\left(\frac{\epsilon}{\mu}\right)\frac{c^2}{a^2}\,\frac{\delta}{2}\cdot(2\pi a) \qquad \left(\text{using } H_c = \sqrt{\frac{\epsilon}{\mu}}\,\frac{c}{a}\right)$$

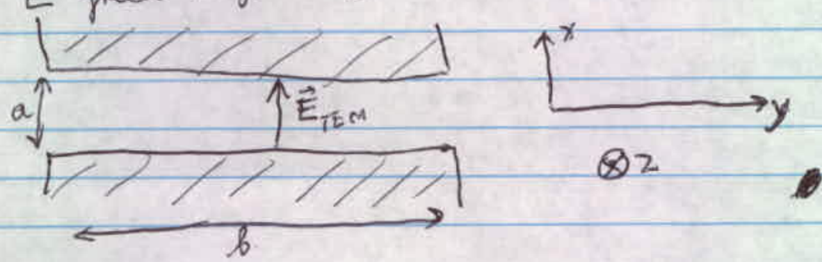
$$= \frac{\mu_c\left(\frac{\epsilon}{\mu}\right)\frac{c^2}{a^2}\,\frac{\delta}{2}\,(2\pi a)}{\left(\frac{\epsilon}{\mu}\right)(2\pi)^2 c^2} = \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}\right) = L_2$$

Adding contribution from outer conductor $= \frac{\mu_c\delta}{4\pi}\left(\frac{1}{b}\right) = L_3$

we get $L = L_1 + L_2 + L_3 = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$

8.3. Again, for TEM Mode, apart from $e^{i(kz-\omega t)}$ factor,

$\vec{E}$ field is given by solution to electrostatic problem



$b \gg a$

$\vec{E}_{TEM} = E_0\,\hat{x}$

$\vec{B}_{TEM} = \sqrt{\mu\epsilon}\;\hat{z}\times\vec{E}_{TEM}$, so $\vec{H}_{TEM} = \sqrt{\frac{\epsilon}{\mu}}\,E_0\,\hat{y} = H_0\,\hat{y}$, say

$\left(H_0 = \sqrt{\frac{\epsilon}{\mu}}\,E_0\right)$

$\vec{S}$ (time averaged) $= \frac{1}{2}(\vec{E}\times\vec{H}^*) = \sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2\,\hat{z}$

so $\vec{P} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2\int_A da = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2 ab = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\,|H_0|^2 ab$

$\gamma = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{2P}\,\frac{1}{2\sigma\delta}\oint_C [\vec{n}\times\vec{H}_{\parallel}]^2 dl$ (Eq. 8.58 of book)

$\longleftrightarrow (|H_0|^2 2b)$ from both surfaces)

$$= \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{|H_0|^2}\,\frac{1}{ab}\,\frac{1}{2\sigma\delta}\,|H_0|^2\,2b$$

$$= \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{a\sigma\delta}$$

Voltage between plates $= E_0 a = \sqrt{\frac{\mu}{\epsilon}}\,H_0\,a$

Total Current $= |\vec{n}\times\vec{H}_{\parallel}|\,b$ ~~[illegible]~~ in each conductor $= H_0\,b$

so $Z_0 = \sqrt{\frac{\mu}{\epsilon}}\,\frac{a}{b}$

Ohmic Losses / unit length $W = \frac{1}{2\sigma}\int |J|^2 da = \frac{1}{2\sigma}\frac{1}{\delta^2} \cdot 2 \cdot |\vec{n}\times\vec{H}_{\parallel}|^2 \left(\frac{\delta}{2}\right)(2b)$

(since $\vec{J}$ is confined to depth $\frac{\delta}{2}$ & length is b for each conductor)

so $W = \frac{1}{2\sigma\delta}|H_0|^2(2b)$

$R = \frac{2W}{I^2} = \frac{1}{\sigma\delta}\frac{|H_0|^2 2b}{|H_0|^2 b^2}$ (from (1)) $= \frac{2}{\sigma\delta b}$

$L = \frac{1}{I^2}\int \frac{\vec{B}\cdot\vec{B}\,dA}{\mu}$ { from space between conductors L_1 / - - - - inside conductors L_2 } $L = L_1 + L_2$

$L_1 = \frac{1}{I^2}\mu \int |H_0|^2 dA = \frac{\mu |H_0|^2 ab}{|H_0|^2 b^2} = \mu\left(\frac{a}{b}\right)$

$L_2 = \frac{1}{I^2}\mu_c \underset{\text{skin current}}{\int} |H_c|^2 dA = \frac{\mu_c \frac{\delta}{2}\cdot 2b\,|H_0|^2}{|H_0|^2 b^2} = \frac{\mu_c \delta}{b}$

$L = \mu\frac{a}{b} + \mu_c\frac{\delta}{b}$

8.4.

(a) TM modes ($\psi = E_z$) $E_z = 0$ at $\rho = R$

soln is $\psi(\rho,\phi) = E_0 J_m(\gamma_{mn}\rho) e^{\pm im\phi}$ (1)

$\gamma_{mn} = \frac{x_{mn}}{R}$ (x_{mn} = n^{th} zero of $J_m(x)$)

$\omega_{mn} = \frac{1}{\sqrt{\mu\epsilon}}\frac{x_{mn}}{R}$ Lowest cutoff frequencies given by

x_{01} (2.405)
x_{11} (3.832)
x_{21} (5.136)
x_{02} (5.520)
x_{12} (7.016)

TE Modes ($\psi = H_z$) $\frac{\partial\psi}{\partial\rho} = 0$ at $\rho = R$

Soln is $\psi(\rho,\phi) = H_0 \gamma_{mn} J'(\gamma_{mn}\rho)$

so $\gamma_{mn} = \frac{x'_{mn}}{R}$ (x'_{mn} = n^{th} zero of $J'_m(x)$)

$\omega_{mn} = \frac{1}{\sqrt{\mu\epsilon}}\frac{x'_{mn}}{R}$ Lowest modes correspond to

x'_{11} (1.848), ~~x'_{21} (3.054), x'_{01} (3.832)~~

x'_{01} (2.45), x'_{21} (3.054), x'_{01} / x'_{11} } (3.832)

(b) $-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_C |\vec{n}\times\vec{H}|^2\, dl$ where $\oint_C$ denotes integral around circumference of guide (Eq. 8.58 of book)

TM Modes $\quad -\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_\lambda}\right)^2 \oint_C \frac{1}{\mu^2\omega_\lambda^2}\left|\frac{\partial\psi}{\partial n}\right|^2 dl \quad$ [Eq. (8.59) of book]

$$= \frac{1}{2\sigma\delta}\,\cancel{\left(\frac{\omega}{\omega_\lambda}\right)^2}\,\frac{\omega^2}{\mu^2\omega_{mn}^4}\oint_C |E_0|^2\gamma_{mn}^2\,|J'(\gamma_{mn}R)|^2\,dl$$

$$= \frac{|E_0|^2}{2\sigma\delta}\,\frac{\omega^2\mu\epsilon}{\mu^2\omega_{mn}^2}\,2\pi R\,|J'(\gamma_{mn}R)|^2 = \frac{|E_0|^2}{\sigma\delta}\,\frac{\epsilon}{\mu}\,\frac{\omega^2}{(\omega_{mn})^2}\,\pi R\,|J'(\gamma_{mn}R)|^2$$

$$P = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{\frac{1}{2}}\epsilon\,|E_0|^2\int_0^R 2\pi\rho\,d\rho\,\left[J_m(\gamma_{mn}\rho)\right]^2$$

(Eq. 8.51 of book)

$$\text{so}\quad \beta_{mn} = -\frac{1}{2P}\frac{dP}{dz} = \sqrt{\frac{\epsilon}{\mu}}\;\frac{1}{\sigma\delta_{mn}}\;\frac{\left(\frac{\omega}{\omega_{mn}}\right)^{\frac{1}{2}}}{\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{\frac{1}{2}}}\;\frac{\pi R\,[J'(\gamma_{mn}R)]^2}{\int_0^R 2\pi\rho\,d\rho\,[J_m(\gamma_{mn}R)]^2} \qquad (2)$$

where we have used the fact that δ at frequency ω is related to δ_{mn} at mode frequency ω_{mn} by $\frac{\delta}{\delta_{mn}} = \left(\frac{\omega_{mn}}{\omega}\right)^{\frac{1}{2}}$ (from definition of δ)

We can write

$$\beta_{mn} = C\,\frac{(\omega/\omega_{mn})^{\frac{1}{2}}}{\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{\frac{1}{2}}} = C'\,\frac{\omega^{3/2}}{\sqrt{\omega^2-\omega_{mn}^2}}$$

This diverges at $\omega = \omega_{mn}$, & has a minimum at $\omega = \sqrt{\frac{3}{2}}\,\omega_{mn}$